INVARIANT MEASURES AND THE EQUICONTINUOUS STRUCTURE RELATION II: THE RELATIVE CASE

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Abstract. In this expository paper we discuss some notions from (abstract) Topological Dynamics. Moreover, we present self-contained simple proofs of the following results. Let $\phi: X \to Y$ be an open extension of minimal flows and suppose that ϕ admits a relatively invariant measure. Then $Q_\phi = E_\phi$, i.e. the relative regionally proximal relation is an equivalence relation. Also, if $E_\phi = R_\phi$ (that is, ϕ has no non-trivial almost periodic factor), then ϕ is weakly mixing.

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INTRODUCTION

In this paper we illustrate an important problem from Topological Dynamics. Our aim is to describe for a general audience a partial solution of this problem. Therefore, this paper will be essentially self-contained. In Sections 1, 2 and 3 we discuss some basic notions (flows, homomorphisms, almost periodic factors of homomorphisms), and we present the main problem to which this paper is devoted: for which homomorphisms ϕ of minimal flows one has $Q_\phi = E_\phi$, i.e. for which ϕ is the relative regionally proximal relation an equivalence relation? We also mention that weak mixing of ϕ implies that $E_\phi = R_\phi$, which means that ϕ has no non-trivial almost periodic factor, and we ask under which additional conditions the converse holds. In Section 4, relatively invariant measures (RIM's) are briefly discussed, and we state that the answer to both problems is affirmative in case ϕ is open and has a RIM. In Section 5, we present the proofs. These results are a generalization of McMahon's paper [5] and were obtained by the second author in his thesis [9] (see also [1]). The "absolute" case of the results to be discussed below, that is, the case that ϕ is the homomorphism of a flow onto the trivial (one-point) flow, is discussed in [8]. Either [8] or the first author's paper [7] can be used as an introduction and motivation for the present paper.

This paper is in final form and no version of it will be submitted for publication elsewhere.

1. FLOWS, HOMOMORPHISMS AND FACTORS

In the sequel of this paper T is a topological group, arbitrary but fixed. A *flow* (also called a T-space with compact Hausdorff phase space, or a compact Hausdorff ttg with acting group T) is a pair $\mathcal{X} := \langle X,\pi\rangle$ where X is a compact Hausdorff space and π is an *action* of T on X. This means that $\pi\colon (t,x) \mapsto tx\colon T \times X \to X$ is a continuous mapping which satisfies the following conditions:

$$ex = x \quad\text{and}\quad t(sx) = (ts)x$$

for all $t,s \in T$ and $x \in X$ (e denotes the unit element of T). We refrain from giving examples; for those, see e.g. [8], 1.3.

If $\mathcal{X} = \langle X,\pi\rangle$ and $\mathcal{Y} = \langle Y,\sigma\rangle$ are flows, then a *homomorphism* from $\mathcal{X}$ to $\mathcal{Y}$ is a continuous mapping $\phi\colon X \to Y$ such that $\phi\circ\pi(t,-) = \sigma(t,-)\circ\phi$ for all $t \in T$; notation: $\phi\colon \mathcal{X} \to \mathcal{Y}$. If $\phi\colon \mathcal{X} \to \mathcal{Y}$ is a homomorphism of flows and $\phi\colon X \to Y$ is a homeomorphism of X onto Y, then ϕ is called an *isomorphism*. A homomorphism $\phi\colon \mathcal{X} \to \mathcal{Y}$ such that $\phi\colon X \to Y$ is a surjection is called an *extension* (of $\mathcal{Y}$). In that case, $\mathcal{Y}$ is also called a *factor* of $\mathcal{X}$, and sometimes ϕ is also called a *factor mapping*. This nomenclature is related to the following observation.

Let $\mathcal{X}$ be a flow and let R be a *closed* invariant equivalence relation in X. Here "invariant" means that R as a subset of $X \times X$ is invariant under the action $(t,(x_1,x_2)) \mapsto (tx_1,tx_2)\colon T\times(X\times X) \to X \times X$ of T on $X \times X$ (it is a straightforward exercise to check that this is, indeed, an action). So $(x_1,x_2) \in R$ implies $(tx_1,tx_2) \in R$ for all $t \in T$. Since R is a closed subset of $X \times X$, the quotient space X/R with the usual quotient topology is a compact Hausdorff space, and as R is invariant an action of T on X/R can unambiguously be defined by

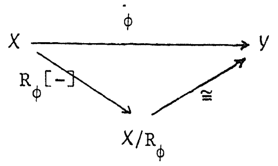
$$tR[x] := R[tx] \quad \text{for } t \in T \text{ and } x \in X.$$

(since the quotient map $R[-]\colon X \to X/R$ is perfect, this action is continuous). Thus, we obtain a flow on X/R, to be denoted by $\mathcal{X}/R$. Clearly, $R[-]\colon \mathcal{X} \to \mathcal{X}/R$ is a factor mapping in the sense defined above, i.e. $\mathcal{X}$ is an extension of $\mathcal{X}/R$ and $\mathcal{X}/R$ is a factor of $\mathcal{X}$.

It is important to observe that *every factor of any flow $\mathcal{X}$ arises in this way*. Indeed, let $\phi\colon \mathcal{X} \to \mathcal{Y}$ be a factor mapping of flows. Then

$$R_\phi := \{(x_1,x_2) \in X\times X : \phi(x_1) = \phi(x_2)\}$$

is a closed invariant equivalence relation in X (invariantness follows from the property that $\phi(tx) = t\phi(x)$ for all $t \in T$ and $x \in X$). It is easy to show that the space X/R_ϕ is homeomorphic with Y (X/R_ϕ is a compact and Y is a Hausdorff space) and that this homeomorphism establishes an isomorphism of flows between X/R_ϕ and Y in such a way that $R_\phi[-]$ corresponds to ϕ :



2. ALMOST PERIODIC EXTENSIONS OF MINIMAL FLOWS

A flow X is called *minimal* whenever it has no proper closed invariant subsets. Equivalently, a flow X is minimal whenever the orbit $Tx(:= \{tx : t \in T\})$ is dense in X for every $x \in X$ (in general, if $P \subseteq T$ and $A \subseteq X$, then $PA := \{sz : s \in P \& z \in A\}$, $tA := \{t\}A$ and $Px := P\{x\}$). By Zorn's lemma and compactness of X, every flow X contains a *minimal subset*, that is, a closed invariant non-empty subset such that the action of the group T, restricted to this subset, defines a minimal flow. For examples, cf. [7],[8].

In the investigations of the structure of minimal flows one often encounters inverse limits. This is the reason for the study of homomorphisms between minimal flows. We shall describe now a couple of notions which are basic for the factorization of certain homomorphisms between minimal flows into an inverse limit of "simple" factors (for more details, cf.[7]).

A homomorphism $\phi: X \to Y$ is called *almost periodic* (or: *equicontinuous*, cf. [7]) whenever)[1]

$$\forall \alpha \in \mathcal{U}_X \; \exists \beta \in \mathcal{U}_X : T\beta \cap R_\phi \subseteq \alpha$$

(in cartesian products of flows we only consider coordinate-wise actions, so $T\beta := \{(tx_1, tx_2) : t \in T \; \& \; (x_1,x_2) \in \beta\}$; note, that $(T\beta) \cap R_\phi = T(\beta \cap R_\phi)$ by invariantness of R_ϕ). So ϕ is almost periodic iff for all $\alpha \in \mathcal{U}_X$ there exists $\beta \in \mathcal{U}_X$ such that the implication "$(x_1,x_2) \in \beta \Rightarrow (tx_1,tx_2) \in \alpha$ for all $t \in T$" is valid only for the points $(x_1,x_2) \in X \times X$ with $\phi(x_1) = \phi(x_2)$. In particular, if T acts uniformly equicontinuous on X, then ϕ is almost periodic. Also, if there is a continuous function $d: R_\phi \to \mathbb{R}$ such that d is a "fibre-wise metric" (i.e. $d|_{\overleftarrow{\phi}[y]\times\overleftarrow{\phi}[y]}$ is a continuous metric on $\overleftarrow{\phi}[y]$ for each $y \in Y$) such that T acts isometrically on fibers (i.e. $\phi(x_1) = \phi(x_2)$ implies $d(tx_1,tx_2) = d(x_1,x_2)$ for all $t \in T$), then ϕ is clearly an almost periodic extension of Y. (If ϕ has this particular property, then ϕ is called an *isometric extension*, cf. [3]. For a

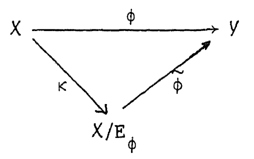
)[1] Here $\mathcal{U}_X$ denotes the (unique) uniformity which is compatible with the topology of X.

generalization of this notion, see [6], 2.4.2 and 2.4.3. Compare also with the continuous IFP's of Section 4 below.)

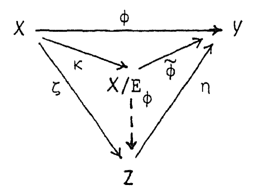
For an arbitrary homomorphism of flows $\phi\colon X \to Y$, let

$$Q_\phi := \bigcap_{\alpha \in \mathcal{U}_X} \overline{T\alpha \cap R_\phi}.$$

It is easily checked that ϕ *is almost periodic iff* $Q_\phi = \Delta_X$, the diagonal in $X \times X$. In general, Q_ϕ is a closed invariant symmetric subset of $R_\phi (\subseteq X \times X)$, and usually it is not transitive, that is, usually Q_ϕ is not an equivalence relation. Let E_ϕ be the smallest closed invariant equivalence relation in $X \times X$ which includes Q_ϕ; E_ϕ is called the *relative regionally proximal relation*). As R_ϕ is a closed invariant equivalence relation which includes Q_ϕ, it follows that $Q_\phi \subseteq E_\phi \subseteq R_\phi$. In particular, this implies that we have the following commutative diagram of homomorphisms:



Here $\kappa\colon X \to X/E_\phi$ is the quotient mapping and $\tilde{\phi}\colon X/E_\phi \to Y$ is unambiguously defined by $\tilde{\phi}(\kappa(x)) := \phi(x)$ for $x \in X$. It is easily checked that $\tilde{\phi}$ is a homomorphism of flows. Although we shall not need it explicitly in the sequel, we mention that the importance of the construction of X/E_ϕ and $\tilde{\phi}$ lies in the fact that $Q_{\tilde{\phi}} = \Delta_{X/E_\phi}$, i.e. $\tilde{\phi}$ is almost periodic. But we shall need, that this construction is "canonical" in the sense that ϕ is, in a well-defined sense, the *maximal almost periodic factor* of ϕ: namely, for every factorization $\phi = \eta \circ \zeta$ of ϕ with η almost periodic one has $E_\phi \subseteq R_\zeta$, which means that the following diagram can be commutatively completed by the dotted arrow:



For details, see [7], 3.9 (where the reader, in turn, will be referred to other literature for the fine details of the proof).

The study of Q_ϕ and E_ϕ plays an important role in abstract Topological Dynamics. In this paper we shall discuss a particular answer to the following questions

(a) Under which conditions is $E_\phi = R_\phi$ (i.e. $\tilde{\phi}$ is an isomorphism); equivalently, when has ϕ no non-trivial almost periodic factor?

(b) Under which conditions is $Q_\phi = E_\phi$, that is, when is Q_ϕ itself already an equivalence relation?

For examples, showing that in (a) and (b) indeed additional conditions are needed, we refer to [5] and the references given there; see also [9]. In [8], similar questions are discussed for the "absolute" case, that is, for the case that $R_\phi = X \times X$ (so Y a singleton). In that situation, the notion of an invariant measure on X turned out to be very useful. In the present, more general, situation, we need the notion of a *relatively invariant measure*.

3. RELATIVELY INVARIANT MEASURES

Let X be a flow and let M(X) denote the set of probability measures on X, endowed with the weak topology. So

$$M(X) := \{\mu \in C_\mu(X)' : \mu \geq 0 \;\&\; \mu(1_X) = 1\},$$

a closed convex subset of the (compact!) unit ball in $C_u(X)'$ with its weak topology. The action of T on X induces an action of T on M(X). This action is given by

$$t\mu(f) := \mu(f\circ\pi(t,-)) = \int_X f(tx)d\mu(x)$$

for $f \in C_u(X)$ and $\mu \in M(X)$. If (via the Riesz representation theorem) an element μ of M(X) is considered as a probability measure (= non-negative regular Borel measure with $\mu(X) = 1$), then the action of T on M(X) is described by

$$(t\mu)(A) := \mu(t^{-1}A)$$

for every Borel subset A of X and $t \in T$. Using a standard compactness argument it is not difficult to show that the mapping $(t,\mu) \mapsto t\mu$: $T \times M(X) \to M(X)$ is continuous. Since it is easily checked that $e\mu = \mu$ and $s(t\mu) = (st)\mu$ for all $\mu \in M(X)$ and $s,t \in T$, it follows that we have, indeed, an action of T on M(X), which defines a flow, denoted by $M(X)$. Observe, that the mapping

$$\delta\colon x \mapsto \delta_x\colon X \to M(X), \text{ where } \delta_x(f) := f(x) \text{ for } f \in C(X),$$

is a topological embedding, and that $\delta_{tx} = t\delta_x$ for all $t \in T$ and $x \in X$. So $\delta\colon X \to M(X)$ is a homomorphism of flows.

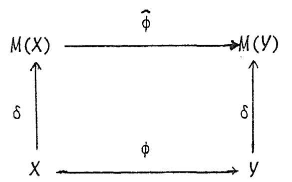
If $\phi\colon X \to Y$ is a homomorphism of flows, then a mapping $\hat{\phi}\colon M(X) \to M(Y)$ is defined by

$$\hat{\phi}(\mu)(f) := \mu(f \circ \phi) \quad \text{for} \quad \mu \in M(X) \quad \text{and} \quad f \in C(X),$$

or, alternatively, by

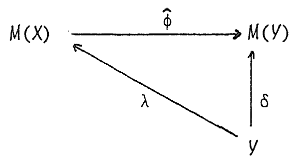
$$(\hat{\phi}\mu)(A) := \mu(\phi^{\leftarrow}[A]) \quad \text{for a Borel set } A \text{ in } Y.$$

It is easy to show that $\hat{\phi}$ is continuous, and a straightforward calculation shows that $\hat{\phi}$: $M(X) \to M(Y)$ is a homomorphism of flows. Moreover, the following diagram commutes:



(We shall make no notational distinction between δ: $X \to M(X)$ and δ: $Y \to M(Y)$, as it is usually clear from the context which mapping is meant)

Now we come to the definition of a *Relatively Invariant Measure* (RIM) for a homomorphism ϕ: $X \to Y$ of flows: this is a homomorphism of flows λ: $Y \to M(X)$ for which the following diagram commutes:



Such a mapping λ is called *a section* for ϕ. Note, that ϕ has a RIM iff for each $y \in Y$ there exists $\lambda_y \in M(X)$ such that

(i) the mapping λ: $y \mapsto \lambda_y$: $Y \to M(X)$ is continuous;
(ii) $\lambda_{ty} = t\lambda_y$ for all $t \in T$ and $y \in Y$;
(iii) the support of λ_y is included in the fiber $\phi^{\leftarrow}(y)$ of y.

Indeed, (i) and (ii) express that λ: $Y \to M(X)$ is a homomorphism of flows, and (iii) is equivalent with the commutativity of the above diagram. (Recall, that *the support* supp μ of $\mu \in M(X)$ is the complement of the largest open set of measure zero, i.e. supp μ is the smallest closed set of measure one; consequently, for an open subset U of X, $\mu(U) = 0$ iff $U \cap \text{supp}\, \mu = \emptyset$.)

REMARK. If we apply this definition to the case that Y is a one-point space, then the obvious homomorphism ϕ: $X \to Y$ has a section λ iff there is a measure $\mu \in M(X)$ (namely, $\mu := \lambda_y$, where y is the unique point of Y) such that $t\mu = \lambda_{ty} = \lambda_y = \mu$ for all $t \in T$, that is, iff X has an invariant measure. Note, that in this case supp $\mu = X = \phi^{\leftarrow}(y)$, provided X is minimal (this is, because for an invariant measure μ the support supp μ is a non-empty, closed invariant subset). In the case of an arbitrary homomorphism of minimal flows $\phi: X \to Y$ which has a RIM λ it is not true that supp $\lambda_y = \phi^{\leftarrow}(y)$ for all $y \in Y$. We will return to this in the next Section. As to the question which homomorphisms admid a RIM, we refer to [4].

Before stating our main theorem we need one more definition. A homomorphism of flows $\phi: X \to Y$ is called *weakly mixing* whenever R_ϕ (as a subflow of $X \times X$) is ergodic, that is, if invariant subsets of R_ϕ are either dense or nowhere dense in R_ϕ. Equivalently, ϕ is weakly mixing whenever for every two open subsets O_1 and O_2 of R_ϕ there is $t \in T$ such that $tO_1 \cap O_2 \neq \emptyset$. Again, this is equivalent to requiring that for every four open subsets U_1, U_2 and V_1, V_2 of X such that $(U_1 \times U_2) \cap R_\phi$ and $(V_1 \times V_2) \cap R_\phi$ are non-empty there exists $t \in T$ such that

$$t(U_1 \times U_2) \cap (V_1 \times V_2) \cap R_\phi \neq \emptyset.$$

The following result generalizes a result from [5], where X was assumed to be metrizable. For more general results and other answers to the questions posed above, see [9] (also, [1]).

<u>THEOREM</u>. *Let $\phi: X \to Y$ be an open homomorphism of minimal flows and suppose that ϕ has a RIM. Then $Q_\phi = E_\phi$. Moreover, if $E_\phi = R_\phi$ then ϕ is weakly mixing.*

The proof of this theorem will be presented in the next section. It should be noted that the second statement in the theorem is the converse of the following, almost trivial, statement (where ϕ need not be assumed to be open, nor is assumed to have a RIM): *if ϕ is weakly mixing then* $E_\phi = R_\phi$.

<u>PROOF</u>. For each $\alpha \in \mathcal{U}_X$, $\overline{T\alpha \cap R_\phi}$ is a closed invariant subset of R_ϕ. Moreover, it has a non-empty interior in R , because α contains an open nbd of Δ_X in $X \times X$. So if ϕ is weakly mixing, then $\overline{T\alpha \cap R_\phi} = R_\phi$ for every $\alpha \in U_X$, hence $Q_\phi = R_\phi$, and therefore $E_\phi = R_\phi$.

4. PROOF OF THE THEOREM

4.1. In this section we consider minimal flows X and Y and a homomorphism $\phi: X \to Y$; Moreover, let $\lambda: Y \to M(X)$ be a section for ϕ. An important notion for the proof of the theorem is that of an *invariant fibre-wise pseudometric* (abbreviated IFP).

A continuous(!) mapping $\rho: R_\phi \to \mathbb{R}^+$ is called an IFP whenever the following conditions are fulfilled:

(i) $\forall y \in Y$: $\rho|_{\phi^{\leftarrow}(y) \times \phi^{\leftarrow}(y)}$ is a pseudometric on $\phi^{\leftarrow}(y)$;

(ii) ρ is invariant on fibers, that is, if $x_1, x_2 \in X$ and $t \in T$, then

$$\phi(x_1) = \phi(x_2) \Rightarrow \rho(tx_1,tx_2) = \rho(x_1,x_2).$$

If ρ is an IFP then let

$$D_\rho := \{(x_1,x_2) \in R_\phi : \rho(x_1,x_2) = 0\}.$$

It is clear that D_ρ is a closed invariant equivalence relation (transitivity of the relation D_ρ follows from the triangle-inequality within fibers). The following simple lemma shows what D_ρ has to do with E_ϕ:

4.2. LEMMA. *Let ϕ be as above. Then for every IFP ρ one has the inclusion* $E_\phi \subseteq D_\rho$.

PROOF. Since D is a closed invariant equivalence relation and, by definition, $D_\rho \subseteq R_\phi$ there exists a unique homomorphism $\bar{\phi}: X/D_\rho \to Y$ such that $\phi = \bar{\phi}\circ\zeta$, where $\zeta: X \to X/D_\rho$ is the quotient mapping. Let $\bar{\rho}: R_{\bar{\phi}} \to \mathbb{R}^+$ be defined by

$$\bar{\rho}(\zeta x_1,\zeta x_2) := \rho(x_1,x_2) \quad \text{for} \quad x_1,x_2 \in X.$$

Then $\bar{\rho}$ is unambiguously defined, and as $\zeta\times\zeta$ is a quotient mapping (all spaces under consideration are compact Hausdorff) it follows that $\bar{\rho}$ is continuous. In addition, it is easily checked that $\bar{\rho}$ defines a metric on each fiber of $\bar{\phi}$, so that, by compactness of the fibers, on each fiber the topology is actually generated by this metric. Since $\bar{\rho}$ is invariant on the fibers of $\bar{\phi}$, it follows that for all $\varepsilon > 0$ and $z_1,z_2 \in X/D_\rho$,

$$\bar{\phi}(z_1) = \bar{\phi}(z_2) \;\&\; \bar{\rho}(z_1,z_2) < \varepsilon \Rightarrow \bar{\rho}(tz_1,tz_2) < \varepsilon \quad \text{for all } t \in T.$$

This means exactly, that $\bar{\phi}$ is almost periodic according to the definition in Section 2 (in fact, $\bar{\phi}$ is an isometric extension). Since E_ϕ defines the maximal almost periodic factor of ϕ, this implies that $E_\phi \subseteq R_\zeta = D_\rho$. □

4.3. We shall now indicate a class C of IFP's such that the set $D(C) := \bigcap\{D_\rho : \rho\in C\}$ has the property that $D(C) \subseteq Q_\phi$. This is sufficient for the proof of the first part of the theorem: indeed, $E_\phi \subseteq D(C)$, by Lemma 4.2, and since $Q_\phi \subseteq E_\phi$, the inclusion $D(C) \subseteq Q_\phi$ implies $Q_\phi = E_\phi = D(C)$.

The construction of the set C is as follows (a number of steps can be done in greater generality, and the results can be sharpened: see Section VII. 3 of [9]). First, let for every subset N of R_ϕ and every point x in X the "section" of N at x be denoted by

$$N[x] := \{x' \in X : (x,x') \in N\}.$$

Obviously, $N[x] \subseteq \phi^{\leftarrow}\phi(x)$ (for $N \subseteq R_\phi$), and if N is closed in R_ϕ then N[x] is closed in $\phi^{\leftarrow}\phi(x)$, hence in X. If N is invariant, then $tN[x] = N[tx]$ for every $t \in T$. We shall see below that if N is a non-empty closed invariant subset of R_ϕ, then $N[x] \neq \emptyset$ for every $x \in X$.

4.4. LEMMA. *Let* N *be a non-empty closed invariant subset of* R_ϕ, *and define the mapping* $\rho_N : R_\phi \to \mathbb{R}^+$ *by* *)

$$\rho_N(x_1,x_2) := \lambda_{\phi(x_1)}(N[x_1] \Delta N[x_2]) \quad \textit{for} \quad (x_1,x_2) \in R_\phi.$$

Then ρ_N *is continuous and, in fact,* ρ_N *is an IFP on* R_ϕ.

PROOF. It is straightforward to check that ρ_N is a pseudometric on each fiber $\phi^{\leftarrow}(y)$ for $y \in Y$ (note that the asymmetry in the definition is just seeming, because for $x_1,x_2 \in \phi^{\leftarrow}(y)$ one has $\lambda_{\phi(x_1)} = \lambda_y = \lambda_{\phi(x_2)}$. Also, it is easy to show, using the various invariantness definitions, that ρ_N is invariant on fibers. So it remains to show, that ρ_N is continuous. This will be done in several steps.
1. For every $x \in X$, the set N[x] is not empty. This is a consequence of invariantness of N: its projection onto the first coordinate is a closed (N is compact!), non-empty ($N \neq \emptyset$!) invariant subset of X, hence all of X (minimality of X). So for every $x \in X$ there is $x' \in X$ with $(x,x') \in N$, that is, $x' \in N[x]$.

Let 2^X denote the space of all closed non-emtpy subsets of X endowed with the Vietoris topology (for the sequel, it is not necessary to know what this means). We have shown, that $N[x] \in 2^X$ for every $x \in X$. We claim:
2. The mapping $x \mapsto N[x]: X \to 2^X$ is *upper semicontinuous*, that is, for every $x \in X$ and every open nbd U of the closed set N[x] in X there exists a nbd V of x in X such that $N[x'] \subseteq U$ for all $x' \in V$. The easy proof by contradiction is left to the reader.
3. If $x_1,x_2 \in X$ then $\lambda_{\phi(x_1)}(N[x_1]) = \lambda_{\phi(x_2)}(N[x_2])$. To prove this, let $\varepsilon > 0$ and let U be an open nbd of $N[x_2]$ in X such that

$$(*) \qquad \lambda_{\phi(x_2)}(U) < \lambda_{\phi(x_2)}(N[x_2]) + \frac{\varepsilon}{2}$$

(regularity of the measure $\lambda_{\phi(x_2)}$). In addition, let U' be an open nbd of the (compact!) set $N[x_2]$ such that $\overline{U'} \subseteq U$ and let $f: X \to [0;1]$ be a continuous function such that $f(x) = 1$ for $x \in \overline{U'}$ and $f(x) = 0$ for $x \notin U$. Then by the inequality above,

$$\lambda_{\phi(x_2)}(f) < \lambda_{\phi(x_2)}(N[x_2]) + \frac{\varepsilon}{2}\,.$$

*) Here Δ denotes the symmetric difference: $A\Delta B = (A\setminus B)\cup(B\setminus A) = (A\cup B)\setminus(A\cap B)$.

Since the mapping $x \mapsto \lambda_{\phi(x)} : X \to M(X)$ is continuous, it follows that there is a nbd V of x_2 in X such that $|\lambda_{\phi(x')}(f)-\lambda_{\phi(x_2)}(f)| < \frac{\varepsilon}{2}$, hence

$$\lambda_{\phi(x')}(f) < \lambda_{\phi(x_2)}(N[x_2]) + \varepsilon$$

for all $x' \in V$. By 2 above, there is a nbd W of x_2 in X such that $N[x'] \subseteq U'$ for all $x' \in W$. Hence for all $x' \in V \cap W$ we have (recall, that $f|_{U'}=1$):

$$\lambda_{\phi(x')}(N[x']) \leq \lambda_{\phi(x')}(f) < \lambda_{\phi(x_2)}(N[x_2]) + \varepsilon.$$

As X is minimal, the point x_1 has a dense orbit, so there is $t \in T$ such that $tx_1 \in V \cap W$, hence

$$\lambda_{\phi(x_1)}(N[x_1]) = \lambda_{\phi(tx_1)}(N[tx_1]) < \lambda_{\phi(x_2)}(N[x_2]) + \varepsilon.$$

This holds for every $\varepsilon > 0$, so $\lambda_{\phi(x_1)}(N[x_1]) \leq \lambda_{\phi(x_2)}(N[x_2])$. Since x_2 has also a dense orbit, a similar proof can be given to establish the reversed inequality.

4. In order to prove that ρ_N is continuous on R_ϕ it is (by the triangle inequality) sufficient to show that for every point $x \in X$ and every $\varepsilon > 0$ there is a nbd V' of x such that

$$\rho_N(x,x') < \varepsilon \quad \text{for all} \quad x' \in V' \text{ with } \phi(x') = \phi(x).$$

First note, that if $\phi(x) = \phi(x')$, then in the right-hand side of the following identity x and x' may be interchanged by 3:

$$\lambda_{\phi(x)}(N[x]\backslash N[x']) = \lambda_{\phi(x)}(N[x])-\lambda_{\phi(x)}(N[x] \cap N[x']),$$

and this shows that $\lambda_{\phi(x)}(N[x]\backslash N[x']) = \lambda_{\phi(x')}(N[x']\backslash N[x])$. However, $\lambda_{\phi(x)} = \lambda_{\phi(x')}$, and using this, we see that

$$(**) \qquad \rho_N(x,x') = \lambda_{\phi(x)}(N[x] \,\Delta N[x']) = 2\lambda_{\phi(x)}(N[x']\backslash N[x]).$$

Now let U and W be as in 3 with x instead of x_2: then for $x' \in W \cap \overleftarrow{\phi}\phi(x)$ we have $N[x'] \subseteq U$, hence by inequalities (*) and identity (**):

$$\rho_N(x,x') \leq 2\lambda_{\phi(x)}(U\backslash N[x]) = 2(\lambda_{\phi(x)}(U)-\lambda_{\phi(x)}N[x]) < 2\varepsilon.$$

This concludes the proof. □

As the family C of IPF's on R_ϕ we shall take

$$C := \{\rho_N : \emptyset \neq N = \overline{TN} \subseteq R_\phi\}$$

(note, that $N = \overline{TN}$ means exactly, that N is closed and invariant).

4.5. <u>First attempt of a proof for the inclusion $D(C) \subseteq Q_\phi$</u>

Suppose that for all $x \in X$ we would have $x \in \operatorname{supp} \lambda_{\phi(x)}$. Then we could prove the desired inclusion as follows (the proof is completely similar to the proof in [8]:

Let $(x_1,x_2) \in D(C)$, $\alpha \in \mathcal{U}_X$, and set $N := \overline{T\alpha \cap R_\phi}$. Then N is a non-empty closed invariant subset of R_ϕ, so $\rho_N \in C$ and by assumption $\rho_N(x_1,x_2) = 0$. Consider an arbitrary nbd U of x_2 such that $U \subseteq \alpha[x_2]$. Then clearly $U \cap \overleftarrow{\phi}\phi(x_2) \subseteq N[x_2]$, so the set $(U\cap\overleftarrow{\phi}\phi(x_2))\backslash N[x_1]$ is a (possibly empty) subset of $N[x_1]\Delta N[x_2]$. However,

$$\lambda_{\phi(x_2)}(N[x_1]\Delta N[x_2]) = \rho_N(x_1,x_2) = 0,$$

so $\lambda_{\phi(x_2)}(U\cap\overleftarrow{\phi}\phi(x_2)\backslash N[x_1]) = 0$. Since we are considering an open set, this implies that

$$(U\cap\overleftarrow{\phi}\phi(x_2)\backslash N[x_1]) \cap \operatorname{supp} \lambda_{\phi(x_2)} = \emptyset,$$

or, equivalently (recall, that $\operatorname{supp} \lambda_{\phi(x_2)} \subseteq \overleftarrow{\phi}\phi(x_2)$):

$$(***) \qquad U \cap \operatorname{supp} \lambda_{\phi(x_2)} \subseteq N[x_1].$$

Since we were assuming that $x_2 \in \operatorname{supp} \lambda_{\phi(x_2)}$ this clearly implies that $x_2 \in N[x_1]$, i.e. $(x_1,x_2) \in N$. So we have shown that $D(C) \subseteq \overline{T\alpha \cap R_\phi}$ for every $\alpha \in \mathcal{U}_X$, which implies $D(C) \subseteq Q_\phi$. □

4.6. The condition that $x \in \operatorname{supp} \lambda_{\phi(x)}$ for all $x \in X$ is rather heavy. It is easy to show that the set $\{x \in X : x \in \operatorname{supp} \lambda_{\phi(x)}\}$ is dense in X, as follows: consider an arbitrary point x_0 in X and $x_1 \in \operatorname{supp} \lambda_{\phi(x_0)}$ (note that for any $\mu \in M(X)$, $\operatorname{supp} \mu \neq \emptyset$). Since the support of $\lambda_{\phi(x_0)}$ is included in the fiber $\overleftarrow{\phi}\phi(x_0)$, it follows that $\phi(x_0) = \phi(x_1)$, so $x_1 \in \operatorname{supp} \lambda_{\phi(x_1)}$. However,

$$tx_1 \in t \operatorname{supp} \lambda_{\phi(x_1)} = \operatorname{supp} t\lambda_{\phi(x_1)} = \operatorname{supp} \lambda_{\phi(tx_1)}$$

and since $\{tx_1 : t \in T\}$ is dense in X, this proves our claim. (If X and Y are metric, a little bit more can be said: cf. [4]).

It is not too difficult to show that if $x \in \operatorname{supp} \lambda_{\phi(x)}$, then ϕ is open at x

([5],2.2, also [9],VII.1.5). The converse is not true (see [5], Example 3.2(3)), but if ϕ is open, then the following lemma can be proved, which is just enough for our purposes.

4.7. LEMMA. *If ϕ is open then there is a dense set of points (x_1,x_2) in R_ϕ with the property that* $x_2 \in \text{supp}\, \lambda_{\phi(x_2)}$.

PROOF. Essential for the proof is the observation that for every open subset W of $X \times X$ such that $W \cap R_\phi \neq \emptyset$ there exist open subsets U and V of X such that $\emptyset \neq (U\times V) \cap R_\phi \subseteq W$ and, in addition, $\phi[U] = \phi[V]$. Assume for the moment that this is true. Every open subset of R_ϕ is of the form $W \cap R_\phi$ with W open in R_ϕ, and if $W \cap R_\phi \neq \emptyset$ one can consider U and V as above. By the observation at the beginning of 4.6, there is a point $x_2 \in V \cap \text{supp}\, \lambda_{\phi(x_2)}$. Now there is $x_1 \in U$ such that $\phi(x_1) = \phi(x_2)$, hence $(x_1,x_2) \in (U \times V) \cap R_\phi \subseteq W \cap R_\phi$. This completes the proof of the lemma.

The proof of the existence of U and V with the desired properties goes as follows: first observe, that there are open sets U' and V' in X such that $\emptyset \neq (U'\times V') \cap R_\phi \subseteq W \cap R_\phi$. Note, that

$$O := \phi[U'] \cap \phi[V'] \neq \emptyset,$$

O is open. Now $U := U' \cap \overset{\leftarrow}{\phi}[O]$ and $V := V' \cap \overset{\leftarrow}{\phi}[O]$ suffice. □

REMARK. The conclusion of the lemma is sufficient for the sequel. Note, that this conclusion can also be drawn if the mapping $\theta: (x,y) \mapsto \phi(x) = \phi(y): R_\phi \to X$ is semi-open (i.e. $\theta[W']$ has a non-empty interior for each non-empty open subset W' of R_ϕ): instead of O, take in the above proof $O' := \text{int}\, \theta(U'\times V') \cap R_\phi$.

4.8. Proof of the inclusion $D(C) \subseteq Q_\phi$ under the assumption of the conclusion of Lemma 4.7

For convenience, we shall write D for D(C).

1. A close inspection of 4.5 shows the following. Starting with any point $x_2 \in X$ and $x_1 \in D[x_2]$ (so that $(x_2,x_1) \in D$, hence by symmetry of D, $(x_1,x_2) \in D$) and any open subset U of $\alpha[x_2]$, where $\alpha \in \mathcal{U}_X$, we have shown that $\{x_1\} \times (U \cap \text{supp}\, \lambda_{\phi(x_2)}) \subseteq \overline{T\alpha \cap R_\phi}$ (this is just formula (***); for the proof of this formula it was not necessary that $x_2 \in U$). Replacing x_2 by x, this means that for every $x \in X$, $\alpha \in \mathcal{U}_X$ and open $U \subseteq \alpha[x]$ we have

$$D[x] \times (U \cap \text{supp}\, \lambda_{\phi(x)}) \subseteq \overline{T\alpha \cap R_\phi}\ .$$

2. Next, we want to show that if U' is a non-empty open subset of X such that $D[U'] = \bigcup \{D[u]: u \in U'\}$ is open — we shall see below that there are

sufficiently many of such sets — and $U' \times U' \subseteq \alpha$, where $\alpha \in \mathcal{U}_X$, then

$$(D[U'] \times U') \cap R_\phi \subseteq \overline{T\alpha \cap R_\phi}.$$

So let (z_1,z_2) be a point of $(D[U']\times U') \cap R_\phi$ and consider an arbitrary basic-open nbd of (z_1,z_2) in $X \times X$, i.e. consider open nbds V_1 of z_1 and V_2 of z_2 in X. Without limitation of generality we may assume that $V_1 \subseteq D[U']$ (here we use that $D[U']$ is open) and that $V_2 \subseteq U'$. Since $(V_1\times V_2) \cap R_\phi$ is an open nbd of (z_1,z_2) in R_ϕ, there is by our assumption (namely, the conclusion of 4.7) a point (w_1,w_2) in $(V_1\times V_2) \cap R_\phi$ such that $w_2 \in \text{supp}\, \lambda_{\phi(w_2)}$. Note, that $w_1 \in V_1 \subseteq D[U']$, so there exists $u \in U'$ such that $(u,w_1) \in D$. First, this implies that $(u,w_1) \in R_\phi$, hence $\phi(u) = \phi(w_1) = \phi(w_2)$ and consequently $w_2 \in V_2 \cap \text{supp}\, \lambda_{\phi(u)}$. Thus,

$$(w_1,w_2) \in D[u] \times (V_2 \cap \text{supp}\, \lambda_{\phi(u)})$$

However, $\{u\} \times V_2 \subseteq U' \times U' \subseteq \alpha$ so $V_2 \subseteq \alpha[u]$. Therefore, we may apply the inclusion of 1 above (with u instead of x and V_2 instead of U). We conclude, that $(w_1,w_2) \in \overline{T\alpha \cap R_\phi}$. Since also $(w_1,w_2) \in V_1 \times V_2$, it follows that

$$(V_1\times V_2) \cap \overline{T\alpha \cap R_\phi} \neq \emptyset.$$

This holds for every basic nbd of (z_1,z_2), so $(z_1,z_2) \in \overline{T\alpha \cap R_\phi}$. This concludes the proof of the claim.

3. The next statement is necessary in order to be able to apply the result of 2 above: for every open subset U of X there is an open subset U' of U such that $D[U']$ is open in X.

In order to prove this, first observe that D is a closed invariant equivalence relation in X, so that we can consider the flow X/D. Let $\kappa := D[\cdot]: X \to X/D$ be the quotient map. Since X is minimal and κ is surjective, it is not difficult to show that X/D is also minimal (for a closed invariant subset A of X/D, the set $\kappa^{\leftarrow}[A]$ is closed and invariant in X, hence all of X). By the lemma below, for each open subset U of X the set $\kappa[U]$ has a non-empty interior $\kappa[U]^0$ in X/D. If we put $U' := U \cap \kappa^{\leftarrow}[\kappa[U]^0]$, then U' is an open subset of U such that the set $D[U'] = \kappa^{\leftarrow}\kappa[U']$ is open in X: indeed, $\kappa[U'] = \kappa[U]^0$ is open in X/D.

4.9. LEMMA. *Let $\kappa: X \to Z$ be a homomorphism of minimal flows. Then κ is semi-open, that is, for each open subset U of X the set $\kappa[U]$ has non-empty interior in Z.*

PROOF. Let U_1 be an open subset of U such that $\overline{U}_1 \subseteq U$. From minimality of X it follows that $X = TU$, so by compactness of X, $X = \bigcup\{tU_1 : t \in F\}$ for a finite subset F of T (note, that tU_1 is open in X since t acts as a homeomorphism of X for every $t \in T$). Consequently, $Z = \bigcup_{t\in F} t\kappa[U_1] = \bigcup_{t\in F} \overline{t\kappa[U_1]}$. By (a finite variant of) Baire's theorem it follows that $\overline{\kappa[U_1]}$ has a non-empty interior (each t acts also as a homeomorphism on Z). Since $\kappa[\bar{U}_1] = \overline{\kappa[U_1]}$ and $\kappa[U] \supseteq \kappa[\bar{U}_1]$, it follows that $\kappa[U]$ as well has non-empty interior. □

In 4.8 we have all ingredients which we need for a proof of the inclusion $D = D(C) \subseteq Q_\phi$ in case ϕ is an open homomorphism.

4.10. THEOREM. *If* ϕ: $X \to Y$ *is a homomorphism of minimal flows and* ϕ *is open*)* *and has a RIM, then* $D(C) = Q_\phi = E_\phi$.

PROOF. We want to show that $D \subseteq Q_\phi$ (here $D := D(C)$). Let $(x_1,x_2) \in D$ and $\alpha \in \mathcal{U}_X$, α symmetric. Fix an open nbd U of x_1 such that $U \times U \subseteq \alpha$. By 4.8(3) there is an open subset U' of U such that D[U'] is open; notice, that $U' \times U' \subseteq U \times U \subseteq \alpha$, so that 4.8(2) is applicable. Since x_2 has a dense orbit in X, there is $t \in T$ such that $tx_2 \in U'$. Now $t(x_1,x_2) \in tD = D$, so $tx_1 \in D[tx_2] \subseteq D[U']$, hence by 4.8(2),

$$t(x_1,x_2) \in (D[U']\times U') \cap R_\phi \subseteq \overline{T\alpha \cap R_\phi}.$$

In particular, it follows that $(x_1,x_2) \in \overline{T\alpha \cap R_\phi}$. This holds for all $\alpha \in \mathcal{U}_X$, hence $(x_1,x_2) \in Q_\phi$. This completes the proof. □

4.11. THEOREM. *If* ϕ: $X \to Y$ *is a homomorphism of minimal flows and* ϕ *is open*)* *and has a RIM, then* ϕ *is weakly mixing iff* $E_\phi = R_\phi$.

PROOF. For the "only if", see the end of Section 3. In order to prove the "if", assume that $E_\phi = R_\phi$ and note that $D = Q_\phi = E_\phi$ by 4.10 (where $D := D(C)$), so that $D = R_\phi$. Let for $i = 1,2$, U_i and V_i be open subsets of X such that $(U_1\times U_2) \cap R_\phi \neq \emptyset$ and $(V_1\times V_2) \cap R_\phi \neq \emptyset$. We have to show that there is $t \in T$ such that $t(U_1\times U_2)\cap(V_1\times V_2) \cap R_\phi \neq \emptyset$, or equivalently, that

$$(*) \qquad (V_1\times V_2) \cap \overline{T(U_1\times U_2) \cap R_\phi} \neq \emptyset.$$

Put $N := \overline{T(U_1\times U_2) \cap R_\phi}$; then N is a closed invariant non-empty subset of R_ϕ, and since $D = R_\phi$ it follows that $\rho_N(x_1,x_2) = 0$ for all points (x_1,x_2) in R_ϕ.

By minimality of X there exists $t_1 \in T$ such that

)* Instead of openness of ϕ one may require any other condition which implies that θ: $(x,y) \to \phi(x) = \phi(y)$: $R_\phi \to X$ is semi-open; cf. the remark at the end of 4.7.

$$W := t_1U_2 \cap V_2 \neq \emptyset.$$

By the observation in 4.6 above there is a point $w \in W \cap \operatorname{supp} \lambda_{\phi(w)}$. In view of the fact that ϕ is open we may assume without limitation of generality that $\phi[U_1] = \phi[U_2]$ and $\phi[V_1] = \phi[V_2]$ (cf. the proof of 4.7). Hence there are $x_1 \in V_1$ and $x_2 \in t_1U_1$ such that $\phi(x_1) = \phi(w) = \phi(x_2)$. Now

$$(\{x_2\}\times W) \cap R_\phi \subseteq t_1(U_1\times U_2) \subseteq N,$$

hence $W \cap \overset{\leftarrow}{\phi}\phi(x_2) \subseteq N[x_2]$. Exactly as in 4.5 this implies (using that $\rho_N(x_1,x_2) = 0$) that

$$W \cap \operatorname{supp} \lambda_{\phi(x_2)} \subseteq N[x_1].$$

In particular, $w \in N[x_1]$ or $(x_1,w) \in N$. Since also $(x_1,w) \in V_1 \times V_2$, this proves that the intersection (*) is non-empty, as wanted. □

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